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ON THE STABILITY OF THE LINING OF A HORIZONTAL OPENING IN A VISCOELASTIC AGEING MEDIUM*

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The stability of a long elastic tube in a viscoelastic medium is studied. Stability conditions, formulated in terms of the characteristics of the tube and the medium, are set up. Such probelms are of interest in studying the stability of underground structures /1-3/. The stability problem for a tube in the case when the medium is elastic was studied in /4/. This paper touches on the investigations in /5,6/.

1. Formulation of the problem. At a depth H from the daylight surface in mountain rock, let there be a working (opening) of circular cross-section of radius R. The rock is considered to be a homogeneous, isotropic, viscoelastic medium filling the half-space. The working is reinforced, i.e., an elastic cylinder is imbedded which is fixed to the material of the rock surrounding the working. The lining is considered to be a homogeneous elastic medium. Far from the ends of the working, plane strain is realized in the rock and the lining. According to /7/, for $H/R > 50$ the problem of determining the state of stress and strain of the lining can be simplified and the lining can be considered as an elastic tube reinforcing a cylindrical hole in a viscoelastic space which is compressed by the uniform forces $p_1 = \gamma H$, $p_2 = \nu(1-\nu)^{-1}\gamma H$ far from the hole, where γ is the specific gravity, and ν is Poisson's ratio of the rock.

Let the viscoelastic medium occupy all three-dimensional space. Let x_1, x_2, x_3 denote the coordinates of points of the medium in a Cartesian coordinate system $Ox_1x_2x_3$. A cylinder $x_1^2 + x_2^2 \leqslant 1$ is cut out of the medium, where the radius can be taken to be equal to unity without loss of generality. A circular elastic tube whose external radius equals unity is inserted into the hole being obtained. At the time $t = 0$ compressive forces of constant intensity p_1 along the Ox_1 axis and p_2 along the Ox_2 axis are applied to the viscoelastic medium at infinity, and a force of intensity g directed perpendicular to the tube axis is applied to the inner surface of the tube. We introduce the cylindrical coordinate system $Or\vartheta x_3$, whose axis Ox_3 coincides with the tube axis, while the polar angle ϑ is measured from the Ox_1 axis. The forces applied to the inner surface of

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the tube are statically equivalent to zero, i.e. (g_1, g_2 are the radial and tangential components of the vector g)

$$\langle g_2\rangle = 0, \langle g_1 \sin\vartheta + g_2 \cos\vartheta\rangle = 0, \quad \langle g_1 \cos\vartheta - g_2 \sin\vartheta\rangle = 0$$

$$\left(\langle f\rangle = \int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta\right)$$

Let $u = (u_1, u_2, u_3)$ be the displacement vector of the medium, and $w = (w_1, w_2, w_3)$ the displacement vector of the tube middle surface in the coordinate system $Or\vartheta x_3$. A plane state of strain is realized in both the tube and the medium under the effect of the forces p, g i.e.

$$u_3 = w_3 = 0, \quad u_i = u_i(t, r, \vartheta), \quad w_i = w_i(t, \vartheta) \quad (i = 1, 2) \tag{1.1}$$

In conformity with the classical Lyapunov definition of the stability of dynamic systems, we call the tube stable if for any $\varepsilon > 0$ there are $\delta_1(\varepsilon) > 0, \delta_2(\varepsilon) > 0$ such that there follows from the inequalities

$$\sup_\vartheta (|g_1(\vartheta)| + |g_2(\vartheta)|) < \delta_1, \quad |p_1 - p_2| < \delta_2$$

the estimate

$$\sup_{t,\vartheta} (|w_1(t, \vartheta)| + |w_2(t, \vartheta)|) < \varepsilon$$

for all $t \geqslant 0, -\pi \leqslant \vartheta \leqslant \pi$.

2. Tube state of stress and strain. We present the equation for the tube deflections under the following assumptions. The tube is an elastic cylindrical shell of infinite length and constant thickness h which is much smaller than its external radius, i.e., $h \ll 1$. The displacements of the tube middle surface are small compared with its thickness. The strain tensor components ε_{jl} are related to the stress tensor components σ_{jl} of the tube in the cylindrcial coordinate system by the equations

$$\varepsilon_{jl} = E_0^{-1}[(1 + \nu_0)\sigma_{jl} - \nu_0\sigma\delta_{jl}] \quad (j, l = 1, 2, 3) \tag{2.1}$$
$$\sigma_{jl} = E_0(1 + \nu_0)^{-1}[\varepsilon_{jl} + \nu_0(1 - 2\nu_0)^{-1}\varepsilon\delta_{jl}]$$

Here E_0 is the elastic modulus, ν_0 is Poisson's ratio of the tube, $\sigma = \sigma_{jj}, \varepsilon = \varepsilon_{jj}$ (summation is performed over repeated subscripts), δ_{jl} are the Kronecker deltas, and we set $\varepsilon_{11} = \varepsilon_{rr}, \varepsilon_{12} = \varepsilon_{r\vartheta}$, etc. The transformations performed below are valid apart from quantities $o(h)$.

We interpret a tube element of unit height and thickness h as a thin elastic curve of a rod /8/. Because of (1.1) this element is in a plane state of strain, i.e., $\varepsilon_{j3} = 0$ for $j = 1, 2, 3$. Moreover, it is considered that Kirchhoff's hypotheses are valid for the tube element (/9/, p. 54) in conformity with which $\varepsilon_{11} = \varepsilon_{12} = 0$. Hence, and from (/10/, p. 26) it follows that

$$\varepsilon_{22} = \varepsilon_{22}^c + (1 - \rho)(\varkappa - \varepsilon_{22}^c); \quad \varkappa = -w_{1,02} + w_{2,01}, \quad \varepsilon_{22}^c = w_{2,01} + w_1 \tag{2.2}$$

Here ρ is the radius of curvature, $\varkappa$ is the additional curvature of the tube element, ε_{22}° is the angle of rotation of the tube element relative to its initial position. The notation $u_{,jl} = \partial^{j+l}u/\partial r^j\partial\vartheta^l$ is used for the derivatives.

Finally, it is assumed that $\varepsilon_{22}^c = o(h)$. Hence, and from the last relationship in (2.2) it follows that

$$w_1 = -w_{2,01} \tag{2.3}$$

The rod radius of curvature ρ is related to the increment in the curvature by $\varkappa = \rho^{-1} - 1$, i.e., for small curvatures of the longitudinal axis $\rho \approx 1 - \varkappa$.

By Kirchhoff's hypothesis $\sigma_{11} = 0$. This means that by virtue of (2.1) the equation $\varepsilon_{11} = -\nu_0(1 - \nu_0)^{-1}\varepsilon_{22}$ holds. Hence, and from (2.1) it follows that

$$\sigma_{22} = E_0(1 - \nu_0^2)^{-1}\varepsilon_{22} \tag{2.4}$$

Let N be a normal force, Q the transverse force, and M the bending moment acting on the tube element, i.e.

$$N = \int_{1-h}^{1} \sigma_{22}\, dr, \quad Q = \int_{1-h}^{1} \sigma_{12}\, dr$$
$$M = \int_{1-h}^{1} \sigma_{22}\left(1 + \frac{h}{2} - r\right) dr$$

Substituting (2.4) and (2.2) here instead of σ_{22}, and taking account of (2.3), we obtain (D is the cylindrical stiffness of the tube)

$$N = E_0h(1-\nu_0^2)^{-1}\varepsilon_{22}^\circ, \quad M = D\varkappa \qquad (2.5)$$
$$(D = E_0h^3[12(1-\nu_0^2)]^{-1})$$

The equilibrium equations for the tube element have the form (/11/, p. 421)

$$\frac{\partial N}{\partial\vartheta} - \frac{Q}{\rho} + q_2 = 0, \quad \frac{\partial Q}{\partial\vartheta} + \frac{N}{\rho} + q_1 = 0, \quad \frac{\partial M}{\partial\vartheta} + Q = 0 \qquad (2.6)$$

The intensity of the total forces applied to the tube element and directed along the r, ϑ axes is denoted by q_1, q_2.

Eliminating N and Q from the equilibrium Eqs. (2.6), we obtain an equation for the bending moment, from which we conclude by taking the second relationship in (2.5) into account that

$$D(\varkappa_{,03} + \varkappa_{,01}) = q_{1,01} - (q_1\varkappa)_{,01} - q_2 \qquad (2.7)$$

Since the tube thickness is small, the displacements of points of the rod longitudinal axis coincide with the displacements of the medium on the boundary with the tube

$$u_1 = w_1, \quad u_2 = w_2, \quad r = 1, \quad -\pi \leqslant \vartheta \leqslant \pi. \qquad (2.8)$$

The deflection of the tube is determined by the last two equations in (2.2), Eqs. (2.7) and the boundary conditions that consist of periodicity in ϑ with period 2π for the functions under consideration and their derivatives. To close these equations it is necessary to find the dependence of the forces q_1 and q_2 on the deflection. This is done below by analyzing the state of stress and strain of the medium.

3. Equations of state of the medium. Let $\sigma_{jl}(t, r, \vartheta)$ and $\varepsilon_{jl}(t, r, \vartheta)$ be the stress and strain tensor components of the medium in the coordinate system $Or\vartheta x_3$. We assume that the mean stress $\sigma = \sigma_{jj}/3$ and the mean bulk strain $\varepsilon = \varepsilon_{jj}/3$ are connected by the relationship

$$\varepsilon = (1-2\nu)\sigma/E \qquad (3.1)$$

where E is the instantaneous elastic modulus and ν is Poisson's ratio of the medium.

The strain and stress tensor deviators e_{jl}, s_{jl} of the medium satisfy the relationship

$$e_{jl} = E^{-1}(1+\nu)(I+K)s_{jl}, \quad e_{jl} = \varepsilon_{jl} - \varepsilon\delta_{jl} \qquad (3.2)$$
$$s_{jl} = E(1+\nu)^{-1}(I-R)e_{jl}, \quad s_{jl} = \sigma_{jl} - \sigma\delta_{jl}$$
$$\left(Ks_{jl} = \int_0^t k(t,\tau)s_{jl}(\tau)\,d\tau, \quad Re_{jl} = \int_0^t r(t,\tau)e_{jl}(\tau)\,d\tau\right)$$

Here I is the unit operator, K is the creep operator, and R is the relaxation operator.

It is assumed that the relaxation kernel is $r(t,\tau) \geqslant 0$ and continuous functions $l_0(t,\tau)$, $l_1(t,\tau)$ and a constant $\beta \in (0, 1)$ exist such that

$$r(t,\tau) = l_0(t,\tau)(t-\tau)^{-\beta} - l_1(t,\tau), \quad 0 \leqslant \tau \leqslant t \qquad (3.3)$$
$$|r| = \sup_t \int_0^t r(t,\tau)\,d\tau < 1$$

Let us transform the equations of state (3.1) and (3.2). By virtue of (3.2) we obtain (λ and μ are the Lamé parameters)

$$\sigma_{jl} = (3\lambda I + 2\mu R)\varepsilon\delta_{jl} + 2\mu(I-R)\varepsilon_{jl} \qquad (3.4)$$
$$\lambda = E\nu[(1+\nu)(1-2\nu)]^{-1}, \quad \mu = E[2(1+\nu)]^{-1}$$

The strain of the medium is planar, i.e., $\varepsilon_{j3} = 0$. Hence, and from (3.4) it follows that $\sigma_{33} = (3\lambda I + 2\mu R)\varepsilon$. Consequently the mean stress is

$$\sigma = [\sigma_{11} + \sigma_{22} + (3\lambda I + 2\mu R)\varepsilon]/3 \qquad (3.5)$$

We substitute its expression in terms of ε from (3.4) for σ into (3.5) and we take account of the expressions for λ and μ. We obtain

$$\sigma_0 = \sigma_{11} + \sigma_{22} = \lambda\nu^{-1}(3I - (1-2\nu)R)\varepsilon \qquad (3.6)$$
$$\varepsilon = (3\lambda)^{-1}\nu(I+K_1)\sigma_0$$

The operator K_1 is defined by the formula $I + K_1 = (I - (1-2\nu)R/3)^{-1}$.

The equality

$$-\sigma_0 + 3\lambda\nu^{-1}\varepsilon = E\,(1+\nu)^{-1}\,R\varepsilon$$

follows from (3.6).

Replacing ε on its left side in conformity with (3.6), we obtain an expression for $R\varepsilon$ which when substituted into (3.4) finally yields

$$(I - R)\,\varepsilon_{jl} = E^{-1}\,(1+\nu)\,[\sigma_{jl} - (\nu I + (1+\nu)\,K_1)\,\sigma_0\delta_{jl}] \tag{3.7}$$

Furthermore, we consider that the elongation, shear, and angle of rotation of the material of the medium are small and can be neglected in the equilibrium equations of the form

$$\sigma_{11,\,10} + \frac{\sigma_{12,\,01}}{r} + \frac{\sigma_{11} - \sigma_{22}}{r} = 0, \quad \sigma_{12,\,10} + \frac{\sigma_{22,\,01}}{r} + \frac{2\sigma_{12}}{r} = 0 \tag{3.8}$$

and the strain compatibility equations

$$\begin{aligned} W(\varepsilon) = 0;\ \ W = {} & \cos^2\vartheta\,\varepsilon_{22,\,20} + \sin^2\vartheta\,(r^{-1}\varepsilon_{22,\,10} + r^{-2}\varepsilon_{22,\,02}) - \\ & 2\sin\vartheta\cos\vartheta\,\frac{\partial}{\partial r}(r^{-1}\varepsilon_{22,\,01}) + \sin^2\vartheta\,\varepsilon_{11,\,20} + \\ & \cos^2\vartheta\,(r^{-1}\varepsilon_{11,\,10} + r^{-2}\varepsilon_{11,\,02}) + 2\sin\vartheta\cos\vartheta\,\frac{\partial}{\partial r}(r^{-1}\varepsilon_{11,\,01}) + \\ & 2\sin\vartheta\cos\vartheta\,(r^{-1}\varepsilon_{12,\,10} + r^{-2}\varepsilon_{12,\,02} - \varepsilon_{12,\,20}) - 2(\cos^2\vartheta - \sin^2\vartheta)\,\frac{\partial}{\partial r}(r^{-1}\varepsilon_{12,\,01}) \end{aligned} \tag{3.9}$$

We substitute the expression for the strain (3.7) into (3.9). We obtain an equation expressing the condition of stress compatibility

$$(1+\nu)\,\Delta\sigma_0 = (I + R_2)\,W(\sigma), \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2} \tag{3.10}$$

The operator R_2 has been introduced into (3.10) using the formula $I + R_2 = (I - K_2)^{-1}$, $K_2 = (1+\nu)\,(1-\nu)^{-1}\,K_1$.

The stresses σ_{jl} are expressed in terms of the Airy function $F = F(t, r, \vartheta)$ by using the equalities

$$\sigma_{11} = r^{-1}F_{,\,10} + r^{-2}F_{,\,02}, \quad \sigma_{22} = F_{,\,20}, \quad \sigma_{12} = -(r^{-1}F_{,\,01})_{,\,10} \tag{3.11}$$

We note that the equilibrium Eqs.(3.8) are satisfied for the stresses defined by (3.11). Substituting (3.11) into (3.10), we obtain

$$\Delta^2 F = 0 \tag{3.12}$$

The boundary conditions for (3.12) have the form (2.8) at the contact points between the tube and the medium, and at infinity are

$$\begin{aligned} 2\sigma_{11} &= -(p_1 + p_2) - (p_1 - p_2)\cos 2\vartheta \\ 2\sigma_{22} &= -(p_1 + p_2) + (p_1 - p_2)\cos 2\vartheta \\ 2\sigma_{12} &= (p_1 - p_2)\sin 2\vartheta, \quad r = \infty \end{aligned} \tag{3.13}$$

Furthermore, it is considered that squares of the elongations, shear, and angle of rotation, i.e.,

$$\begin{aligned} \varepsilon_{11} &= u_{1,10}, \quad \varepsilon_{22} = r^{-1}(u_{2,01} + u_1) \\ 2\varepsilon_{12} &= u_{2,10} + r^{-1}(u_{1,01} - u_2) \end{aligned} \tag{3.14}$$

can be neglected in considering the relation between the strains and the displacement of the medium.

We set $F = F_0 + F_1$. Here F_0 satisfies (3.12), the boundary conditions (3.13), and the zero boundary conditions (2.8). The function F_1 satisfies (3.12), the boundary condition (2.8), and the zero boundary conditions (3.13). We let σ_{jl}^k denote the stress defined by (3.11) for $F = F_k, k = 0, 1$. It is clear that

$$q_1 = g_1 - \sigma_{11}(t, 1, \vartheta), \quad q_2 = g_2 + \sigma_{12}(t, 1, \vartheta) \tag{3.15}$$

Consequently, to determine q_j it is sufficient to find F_j because $\sigma_{jl} = \sigma_{jl}^{\circ} + \sigma_{jl}^{1}$.

4. Construction of F_0.

We seek F_0 in the form

$$F_0 = (c_1 r^2 + c_2)\ln r + c_3 r^2 - c_4 + (c_5 r^4 + c_6 r^2 + c_7 + c_8 r^{-2})\cos 2\vartheta \tag{4.1}$$

where thetime functions $c_j(t)$ are to be determined. We substitute (4.1) into (3.11). Then by virtue of the boundary conditions (3.13) and the boundedness of σ_{jl}^{c} for $r \geqslant 1$, we obtain

$$c_1 = c_5 = 0, \quad c_3 = -(p_1 + p_2)/4, \quad c_6 = (p_1 - p_2)/4 \tag{4.2}$$

Hence, it follows from (4.1) and (3.11) that

$$\sigma_{11}^{\circ} = -(p_1+p_2)/2 + c_2r^{-2} - [(p_1-p_2)/2 + 4c_7r^{-2} + 6c_8r^{-4}]\cos 2\vartheta \quad (4.3)$$
$$\sigma_{22}^{\circ} = -(p_1+p_2)/2 - c_2r^{-2} + [(p_1-p_2)/2 + 6c_8r^{-4}]\cos 2\vartheta$$
$$\sigma_{12}^{c} = -2[(p_1-p_2)/4 + c_7r^{-2} + 3c_8r^{-4}]\sin 2\vartheta$$

To determine the remaining functions $c_j(t)$ we use the zero boundary conditions (2.8). We first express $\varkappa$ in terms of the strain. Because of (3.14), we will have $u_1 = r\varepsilon_{22} - u_{2,01}$. Differentiating this eauation with respect to r and using (3.14) we obtain

$$\varepsilon_{11} = \partial(r\varepsilon_{22})/\partial r - u_{2,11}, \quad u_{2,10} = 2\varepsilon_{12} - r^{-1}(u_{1,01} - u_2)$$

We differentiate the second of these relationships with respect to ϑ and we add it to the first. By virtue of the second equation in (2.2) we obtain

$$\varkappa = \varepsilon_{22} + \varepsilon_{22,10} - 2\varepsilon_{12,01} - \varepsilon_{11}, \quad r = 1 \quad (4.4)$$

Since the boundary conditions (2.8) are zero in the case under consideration, then $\varkappa = 0$ for $r = 1$, i.e., taking (4.4) into account we obtain

$$\varepsilon_{22} = 0, \quad \varepsilon_{22,10} - 2\varepsilon_{12,01} - \varepsilon_{11} = 0, \quad r = 1 \quad (4.5)$$

We now determine the strain corresponding to the stresses (4.3) by means of (3.7) and substitute them into (4.5). We obtain an expression for c_2, c_7, c_8. For $r = 1$ we have by virtue of (4.3)

$$\sigma_{11}^{\circ} = -[A(t)(p_1+p_2) + B(t)(p_1-p_2)\cos 2\vartheta] \quad (4.6)$$
$$\sigma_{12}^{c} = [2B(t) + {}^3/_2](p_1-p_2)$$
$$A(t) = 1 - [3\nu I + (1-2\nu)R][3I - (1-2\nu)R]^{-1}\cdot 1$$
$$2B(t) = -1 + {}^1/_2[3I + 4(4\nu I - (1-2\nu)R)]^{-1}\cdot 1$$

It is seen from (4.6) that for $p_1 = p_2$ and $g = 0$ there is just the normal force N balancing the external pressures in the tube.

5. Determination of the components σ_{jl}^1. From the formula for the complex representation of the stress (/12/ p. 136) it follows that

$$\sigma_{11}^1 - i\sigma_{12}^1 = \varphi' + \overline{\varphi}' - e^{2i\vartheta}(\bar{z}\varphi'' + \psi'), \quad |z| \geqslant 1 \quad (5.1)$$

Here $\varphi(t,z)$, $\psi(t,z)$ are functions of time $t \geqslant 0$ and the complex variable $z = re^{i\vartheta}$, the prime denotes the derivative with respect to z, and the upper bar the complex conjugate. The functions φ and ψ are analytic in z for each fixed t. Furthermore, by modifying the derivation of the Kolosov formula (/12/, p. 327) we obtain that the following equality holds in the circle $\Gamma = \{z: |z| = 1\}$

$$\varkappa_1(I - K_3)\varphi - z\overline{\varphi}' - \overline{\psi} = g \quad (5.2)$$

Here

$$g(t,z) = E(1+\nu)^{-1}(I - R)(u_1 - iu_2)e^{i\vartheta} \quad (5.3)$$
$$\varkappa_1 = 3-4\nu = (\lambda+3\mu)(\lambda+\mu)^{-1}, \quad K_3 = 4(1-\nu)(3-4\nu)^{-1}K_1$$

We introduce the operator R_3 by means of the formula $(I - R_3) = (I - K_3)^{-1}$. There results from (5.1) and the well-known results in (/12/, p. 315-317) that

$$\varphi(t,z) = -\frac{1}{2\pi i\varkappa_1}(I + R_3)\int_\Gamma \frac{g(t,s)\,ds}{s-z}, \quad |z| > 1 \quad (5.4)$$
$$\psi(t,z) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{g(t,s)}}{s-z}ds - \frac{1}{z}\varphi' + \psi(t,\infty)$$
$$\psi(t,\infty) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{g(t,s)}}{s}ds$$

To make the functions φ and ψ specific, we introduce the following function on Γ:

$$V(t,z) = -(u_{2,01} - iu_2)e^{i\vartheta}, \quad |z| = 1$$

We note that $g = E(1+\nu)^{-1}(I - R)V$ on the basis of (5.3). Substituting the Fourier-series expansion for the functions $u_2(t,\vartheta)$

$$w_2(t, \vartheta) = \frac{1}{2}a_0(t) + \sum_{n=1}^{\infty}[a_n(t)\cos n\vartheta - b_n(t)\sin n\vartheta] \tag{5.5}$$

into the boundary condition (2.8), we obtain that on Γ

$$u_1 = a_1\sin\vartheta - b_1\cos\vartheta + \sum_{n=2}^{\infty} n(a_n\sin n\vartheta - b_n\cos n\vartheta) \tag{5.6}$$

$$u_2 = \frac{1}{2}a_0 + a_1\cos\vartheta + b_1\sin\vartheta + \sum_{n=2}^{\infty}(a_n\cos n\vartheta + b_n\sin n\vartheta)$$

The components in the right side of each of the relationships (5.6) outside the infinite sums describe the displacement of the boundary of the medium as a rigid whole: counter-clockwise rotation around the Ox_3 axis through an angle $a_0/2$, displacement by a distance $-b_1$ along the Ox_1 axis, and displacement a distance a_1 along the Ox_2 axis.

In the case under consideration the boundary conditions (3.13) are zero, hence they undergo no rigid displacements of the medium as a whole, i.e., the whole medium will be displaced exactly as is its boundary. However, when considering the strain of the medium, its displacement as a rigid whole is of no interest, hence we can set $a_0 = a_1 = b_1 = 0$. Taking account of these equalities we obtain

$$V(t, z) = -\frac{1}{2}\sum_{n=2}^{\infty}[(n-1)c_n z^{n-1} + (n+1)c_n z^{-n-1}]$$

$$c_n(t) = b_n(t) + ia_n(t)$$

It hence follows from the theorem on residues and relationships (5.3) and (5.4) that

$$\varphi(t, z) = -S\sum_{n=2}^{\infty}(n-1)\bar{c}_n z^{-n-1} \tag{5.7}$$

$$\psi(t, z) = S_1\sum_{n=2}^{\infty}(n-1)\bar{c}_n z^{-n-1} - S\sum_{n=2}^{\infty}(n^2-1)\bar{c}_n z^{-n-1} + \psi(t, \infty)$$

$$S = \mu\varkappa_1^{-1}(I + R_3)(I - R), \quad S_1 = \mu(I - R)$$

We substitute (5.7) into (5.1) and we separate real and imaginary parts. We finally obtain that on Γ (i.e., for $|z| = 1$)

$$\sigma_{11}^1(t, \vartheta) = \sum_{n=2}^{\infty}(n^2-1)(S - S_1)(b_n\cos n\vartheta - a_n\sin n\vartheta) \tag{5.8}$$

$$\sigma_{12}^1(t, \vartheta) = \sum_{n=2}^{\infty}(n^2-1)(S_1 - S)(a_n\cos n\vartheta + b_n\sin n\vartheta)$$

Formulas (4.6) and (5.8) determine the action of a tangential force of intensity σ_{12} and of normal pressure of intensity $-\sigma_{11}$ on the tube fromthe viscoelastic medium.

The desired forces q_1 and q_2 in (2.7) are

$$q_1 = g_1 - \sigma_{11}^c - \sigma_{11}^1, \quad q_2 = g_2 - \sigma_{12}^c + \sigma_2^1 \tag{5.9}$$

where σ_{11}^c and σ_{11}^1 have the form (4.6), while σ_{11}^1 and σ_{12}^1 are given by (5.8).

6. Tube stability conditions.

We set up the stability conditions under the additional assumption

$$\sigma_{11}^c \gg \sigma_{11}^1 + g_1 \tag{6.1}$$

Taking (6.1) into account we write the equation for the deflections (2.7) and (5.9) in the form

$$D(w_{2,06} + 2w_{2,04} + w_{2,02}) + \partial[\sigma_{11}^c(w_{2,01} + w_{2,03})]/\partial\vartheta = q_{1,01} - q_2 \tag{6.2}$$

The boundary conditions for (6.2) consist of the values of the function w_2 and its derivatives to fifth order, inclusive, being equal at the points $-\pi$ and π.

Let us take a certain integer $n \geqslant 2$. We multiply both sides of (6.2) by $\cos n\vartheta$ and we integrate with respect to ϑ between the limits $-\pi$ and π. Taking account of the orthogonality of the system of functions $\sin n\vartheta$ and $\cos n\vartheta$ and the expansions (5.5), (5.8), (5.9), we obtain

$$\pi(n^2-1)J(t, n)a_n(t) = G(n) + {}^1/_2 nB(t)\pi(p_1 - p_2)[a_{n+2}(n-2)((n-2)^2-1) - a_{n-2}(n-2)((n-2)^2-1)] \tag{6.3}$$

$$G(n) = \int_{-\pi}^{\pi}(g_2\cos n\vartheta + ng_1\sin n\vartheta)\,d\vartheta$$

$$J(t, n) = Dn^2(n^2-1) - A(t)(p_1 + p_2)n^2 - (S_1 - S) + (S_1 + S)n$$

It is clear that

$$|G(n)| \leqslant \sqrt{\pi}(n+1)|g| \tag{6.4}$$

$$|g| = \left(\int_{-\pi}^{\pi} g_1^2\, d\vartheta\right)^{1/2} + \left(\int_{-\pi}^{\pi} g_2^2\, d\vartheta\right)^{1/2}$$

We furthermore assume that $(I + R_3)(I - R) = I - R_5$, where the kernel $r_5(t, \tau)$ of the operator R_5 satisfies conditions of the form (3.3). We introduce the notation

$$A_n(t) = \max_\tau |a_n(\tau)|, \quad B_n(t) = \max_\tau |b_n(\tau)|, \quad 0 \leqslant \tau \leqslant t$$

$$\lambda(n, r, r_5) = Dn^2(n^2 - 1) + \mu(1 - |r|)(n - 1) + \mu\varkappa_1^{-1}(1 - |r_5|)(n + 1)$$

$$\lambda_1(r, r_5) = \min_n n^{-2}\lambda(n, r, r_5), \quad n \geqslant 2$$

It is clear that

$$\left|\int_0^t r(t, \tau)a_n(\tau)\,d\tau\right| \leqslant |r|A_n(t), \quad \left|\int_0^t r_5(t, \tau)a_n(\tau)\,d\tau\right| \leqslant |r_5|A_n(t)$$

Consequently, taking account of (6.3) and (6.4), we have

$$\Lambda(n)A_n(t) \leqslant L(A_n) \tag{6.5}$$

$$\Lambda(n) = \lambda(n, r, r_5) - n^2 A(t)(p_1 + p_2)$$

$$L(A_n) = [\sqrt{\pi}(n - 1)]^{-1}|g| + nB(t)|p_1 - p_2|[2(n^2 - 1)]^{-1}[\beta(n + 2)A_{n+2} + \beta(n - 2)A_{n-2}], \quad n \geqslant 2; \ \beta(n) = n(n^2 - 1)$$

We impose two constraints on the parameters of the problem

$$(p_1 + p_2)A(t) < \lambda_1(r, r_5) \tag{6.6}$$

$$|B(t)(p_1 - p_2)|\sup_n \beta(n + 2)[\Lambda^{-1}(n)\beta_1(n)\chi(n) + \Lambda^{-1}(n + 4)\beta_1(n + 4)] < 2, \ n \geqslant 0; \ \beta_1(n) = n(n^2 - 1)^{-1} \tag{6.7}$$

Here $\chi(n) = 0$ for $n = 0, 1$ and $\chi(n) = 1$ for $n \geqslant 2$.
Because of (6.6) there is a constant $c > 0$ such that

$$\Lambda(n) \geqslant cn^4 \tag{6.8}$$

Now we sum both sides of (6.5) with respect to n. Taking account of (6.6)-(6.8), we can show that a constant $c_1 > 0$ exists such that

$$\sum_{n=2}^{\infty} A_n(t) \leqslant c_1|g| \tag{6.9}$$

The sum of the functions $B_n(t)$ is estimated analogously

$$\sum_{n=2}^{\infty} B_n(t) \leqslant c_1|g| + c_2|B(t)(p_1 - p_2)| \tag{6.10}$$

Therefore

$$|w_1(t, \vartheta)| \leqslant \sum_{n=2}^{\infty}[A_n(t) + B_n(t)] \tag{6.11}$$

To estimate w_2 we require that

$$|B(t)(p_1 - p_2)|\sup_n \beta_2(n + 2)[\Lambda^{-1}(n)\beta_1(n)\chi(n) + \Lambda^{-1}(n + 4)\beta_1(n + 4)] < 2, \ n \geqslant 0 \tag{6.12}$$

$$\beta_2(n) = n^2 - 1$$

On satisfying conditions (6.12), we deduce, as for (6.9) and (6.10), that

$$\sum_{n=2}^{\infty} n(A_n + B_n) \leqslant c_1|g| + c_2|B(t)(p_1 - p_2)| \tag{6.13}$$

This means that

$$|w_2(t, \vartheta)| = |w_{1,\vartheta}| \leqslant \sum_{n=2}^{\infty} n(A_n + B_n) \tag{6.14}$$

By virtue of (6.11), (6.14) we thereby establish

Theorem 1. Let the assumptions formulated above be satisfied. Then the tube is stable when conditions (6.6), (6.7), (6.12) are satisfied.

The stability conditions can be formulated in other terms depending on the limit behaviour of the kernels r and r_5.

Theorem 2. Let functions $r^c(t,\tau)$ and $r_5^c(t,\tau)$ exist such that

$$\lim_{T\to\infty}\sup_{t\geqslant T}\int_T^t [\,|r(t,\tau)-r^\circ(t,\tau)|+|r_5(t,\tau)-r_5^\circ(t,\tau)|]\,d\tau=0$$

$$|r^c|<1,\quad |r_5^c|<1$$

Then the tube is stable when conditions (6.6), (6.7) and (6.12) are satisfied everywhere in which r° should replace r and r_5° should replace r_5.

Remark. 1°. Let $p_1=p_2=p$. Requirements (6.7) and (6.12) are satisfied here. Then the tube is stable for $p<[1-A(t)]^{-1}\lambda_1(r,r_5)$ under the conditions of Theorem 1, and for $p<[1+A(t)]^{-1}\lambda_1(r^\circ,r_5^\circ)$ under the conditions of Theorem 2 (the function $A(t)$ is defined in (4.6)).

2°. Let $p_1=p_2=p$, and the equations of state of the medium have the simpler form

$$\sigma_{jl}=E(1+\nu)^{-1}(I-R)[\varepsilon_{jl}+3\nu(1-2\nu)^{-1}\varepsilon\delta_{jl}]$$

Then both the formulation and the proof are simplified, in which it is necessary to set $K_3=R_3=0$ everywhere. In particular, in this case

$$\sigma_{11}^\circ=-p[1+\mu(\lambda+\mu)^{-1}r^{-2}],\quad \sigma_{12}^c=0$$
$$\sigma_{22}^c=-p[1-\mu(\lambda+\mu)^{-1}r^{-2}]$$

Under the conditions of Theorem 1 the tube is stable for

$$p<(\lambda+\mu)(\lambda+2\mu)^{-1}\lambda_1(r,0) \tag{6.15}$$

and under the conditions of Theorem 2 the quantity r in (6.15) is replaced by r°.

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